Evaluating Value-at-Risk Models with Desk-Level Data

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We present new evidence on disaggregated profit and loss (P/L) and value-at-risk (VaR) forecasts obtained from a large international commercial bank. Our data set includes the actual daily P/L generated by four separate business lines within the bank. All four business lines are involved in securities trading and each is observed daily for a period of at least two years. Given this unique data set, we provide an integrated, unifying framework for assessing the accuracy of VaR forecasts. We use a comprehensive Monte Carlo study to assess which of these many tests have the best finite-sample size and power properties. Our desk-level data set provides importance guidance for choosing realistic P/L-generating processes in the Monte Carlo comparison of the various tests. The conditional autoregressive value-at-risk test of Engle and Manganelli (2004) performs best overall, but duration-based tests also perform well in many cases.

Key words: risk management; backtesting; volatility; disclosure

History: Received July 30, 2007; accepted October 24, 2008, by John Birge, special issue editor. Published online in Articles in Advance.

1. Introduction

In the financial services industry, a primary concern of money managers is the ongoing level of risk in their portfolios. For decades, the textbook measure of portfolio risk was the standard deviation or “volatility.” However, by the 1990s banks began widespread adoption of value at risk (VaR) as an internal definition of portfolio risk, where the VaR is defined as the lower end of a 99% confidence interval. It is now arguably the single most prevalent financial risk measure used in banking and is becoming increasingly common even in nonfinancial firms (see, for example, Jorion 2006 for an extensive overview of VaR).

The widespread use of VaR as an internal measure of risk was given regulatory recognition under the 1996 Market Risk Amendment to the Basel Accord (Basil Committee on Banking Supervision 1996). Under this system, banks are allowed to have their regulatory required capital based on the bank’s own internal VaR forecasts. Although VaR began as a way to measure risk, it is now also used as a management tool. A large bank has a fixed amount of capital that can be allocated by management to traders. To manage overall risk, each trader is typically given a trading limit of some kind. Those trading limits are now typically based on the trader’s portfolio VaR. To a certain extent, traders and portfolio managers even use VaR to guide portfolio choice. If a manager observes VaR increasing, it may signal an undesired increase in risk and trigger the closing of a position. For all of these reasons, both financial services firms themselves as well as Federal Reserve and FDIC regulators have an enormous incentive to make sure that bank’s VaR forecasts are accurate.

In this paper, we provide an integrated, unifying framework for assessing the accuracy of VaR forecasts. Our approach includes the existing tests proposed by Christoffersen (1998) and Christoffersen and Pelletier (2004) as special cases. In addition, we describe some new tests, which are suggested by our framework. To provide guidance as to which of these many tests have the best finite-sample size and power properties, we conduct a thorough Monte Carlo horserace where the profit and loss (P/L)-generating processes are based on four real P/L series.

We obtained the actual daily profit and loss generated by four separate business lines or “desks” from a large, international commercial bank. Each of the business line’s P/L series is observed daily for a period of more than two years. Although of interest in its own
right, the desk-level data set also provides important
guidance for choosing realistic P/L-generating pro-
cesses in our Monte Carlo comparison of backtesting
methods.

In addition to the daily P/L data, we obtained the
corresponding daily, one-day-ahead VaR forecasts
computed using historical simulation. For each busi-
ess line within the bank, and for each day, the VaR
forecasts are estimates of the 1% lower tail. Our data
set complements that of Berkowitz and O’Brien (2002),
which obtained daily bank-wide P/L and VaR data,
but who were not able to obtain any information on
separate business lines within the same bank. In
recent work, Perignon et al. (2007) and Perignon and
Smith (2008) also analyze bank-level VaRs. They find
that one-day-ahead VaR based on historical simula-
tion is the industry standard. For the longer horizons
required by supervisory bodies, such as 10 days ahead,
banks typically simply use the square root of 10 to
scale the one-day-ahead VaR.

Our umbrella framework for testing the accuracy of
a VaR model is based on the observation that the VaR
forecast is a (onesided) interval forecast. Violations—
the days on which portfolio losses exceed the VaR—
should therefore be unpredictable. In particular, the
violations form a martingale difference sequence.
The martingale hypothesis has a long and distinguished
history in economics and finance (see Durlauf 1991).

As a result of this extensive toolkit, we are able
to cast all existing methods of evaluating VaR under
a common umbrella of martingale tests. This imme-
diately suggests several testing strategies. The most
obvious is a test of whether any of the autocovari-
ances are nonzero. The standard approach to test for
un correlatedness is by estimating the sample autocovari-
cances or sample autocorrelations. In particular, we
suggest the well-known Ljung-Box test of the viola-
tion sequence’s autocorrelation function.

The second set of tests are inspired by Campbell and
Shiller (1987) and Engle and Manganelli (2004). If the
violations are a martingale difference sequence, then
they should be un correlated by any transformation of
the variables available when the VaR is computed. It
suggests a regression of the violations/nonviolations
on their lagged values and other lagged variables such
as the previous day’s VaR.

A third set of tests are adapted from Christoffersen
and Pelletier (2004), who focus on hazard rates and
durations. These tests are based on the observation
that the number of days separating the violations (i.e.,
the durations) should be unpredictable.

Last, a fourth set of tests is taken from Durlauf
(1991). He derives a set of tests of the martingale
hypothesis based on the spectral density function. This
approach has several features to commend it. Unlike
variance ratio tests, spectral tests have power against
any linear alternative of any order. Spectral den-
sity tests have power to detect any second moment
dynamics. Variance ratio tests are typically not con-
sistent against all such alternatives.

Because the violation of the VaR is, by construction,
a rare event, the effective sample size in realistic risk
management settings can be quite small. It follows
that we cannot rely on the asymptotic distribution
of the tests to conduct inference. We instead rely on
Dufour’s (2006) Monte Carlo testing technique, which
yields tests with exact level, irrespective of the sample
size and the number of replications used. Our results
suggest that the conditional autoregressive value-at-
risk (CaViaR) test of Engle and Manganelli (2004) per-
forms best overall, but that duration-based tests also
perform well in many cases.

This paper proceeds as follows. In §2, we discuss
the use of VaR as a managerial and operational tool
within financial services firms. In §3, we present the
actual desk-level daily P/Ls and VaRs from several
business lines from a large international bank. Sec-
tion 4 gives an overview of existing methods for
backtesting VaR estimates, and it suggests a few new
approaches as well. Section 5 presents the results of
a detailed horserace among the methods in terms of
size and power properties in the finite sample. In §6,
we report the results from applying the test to our
unique desk-level data sample, and we also assess the
ability of VaRs to forecast P/L volatility.

2. VaR as a Managerial Tool
The one-day 1% VaR of a given portfolio is a dollar
amount, such that daily portfolio loss will be worse
than the VaR only 1% of the time. This provides a sim-
ple one-dimensional snapshot of the downside risk
of the profit and loss distribution. This simplicity is
a key reason for its widespread adoption, although
it clearly represents a somewhat limited amount of
information about the P/L distribution. A key advan-
tage of VaR is that it does not rely on any assumptions
of asset return normality.

2.1. Risk Controls Using VaR
A typical large commercial or investment bank will
have its trading operations organized in a set of trad-
ing desks. The organization typically includes a desk
for equities, one for currencies, one for fixed-income,
and one for derivatives. The risk management team in
the bank has to monitor in real time that each trading
desk stays within the predefined risk limits imposed
by management.

Before the advent of VaR, such risk limits were typ-
ically set in the form of notional limits and/or stop-
loss limits. Examples of notional limits include a
maximum allowed amount invested in a particular
currency, in bonds of a certain maturity, or in equities

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from a particular industry. Such notional limits are problematic for several reasons, including the fact that they are not easily comparable across asset classes.

The stop-loss limits instead force the desk to unwind positions when the accumulated loss on a position has reached a preset level. Stop-loss limits are comparable across assets, but they suffer from being backward-looking in nature, only measuring risk once the loss is realized.

Using VaR limits as risk controls has the advantage that a forward-looking risk measure is used. The VaR is forward-looking by definition because it reports for example the maximum loss over the next day with 99% probability. Empirical evidence on the forward-looking nature of VaR estimates is provided in Taylor (2005), who shows how VaR estimates can be used to forecast future volatility. Furthermore, VaR limits are comparable across asset classes because the VaR of a position reflects both the notional size of the position as well as the risk per dollar invested. Blanco and Blomstrom (1999) provide a more detailed discussion of the advantages of VaR-based risk limits.

2.2. VaR-Based Portfolio Choice

VaR-based risk controls as described above form a passive use of VaR. That is, they do not inform the trading desk how to optimally trade when facing VaR risk limits nor do they tell management how to set optimal VaR-based limits.

In theory, VaR can be used for portfolio choice if it is used as a constraint for the optimal investment policy. For example, optimal portfolio weights can be found by maximizing the expected return or expected utility of terminal wealth subject to a maximum VaR. Basak and Shapiro (2001) have argued on theoretical grounds against the use of VaR as a portfolio optimization constraint; it can encourage excessive risk taking because VaR does not penalize extreme losses. They recommend using an expected shortfall, also known as a CVaR constraint, instead. Alexander and Baptista (2004) also compare the use of VaR and CVaR constraints in portfolio selection and find that CVaR generally dominates VaR except in the absence of a risk-free asset.

However, these critiques of VaR as a portfolio optimization constraint have since been challenged in Cuoco et al. (2008). They show that if the VaR is recomputed dynamically using available information, as is realistic, then the risk exposure of a trader using VaR constraints is always lower than the unconstrained trader.

2.3. Regulatory Uses

Under U.S. banking regulations, commercial banks engaged in trading risky financial assets are required to maintain a minimal level of safe assets as a cushion against unforeseen risk. Since the 1996 Market Risk Amendment to the Basel Accord, qualifying banks can opt to set this required capital level as a function of their VaR. Banks are permitted to use their own internal models to calculate their VaR. Backtesting has been given further relevance by its prominence in the discussion of the Supervisory Review Process (the Second Pillar) in Basel II (Basel Committee on Banking Supervision 2004).

Although no particular technique for backtesting is currently suggested in the Basel Accord, Lopez (1999) notes that the required capital for market risk includes a multiplier based on the unconditional number of VaR violations. In this paper, we develop backtesting techniques that assess both the unconditional VaR and bunching in VaR violations. The results of our horse race show the potential for supervisor endorsement of these more advanced backtesting techniques.

3. Desk-Level P/L and VaR at a Commercial Bank

We collected the actual daily P/L generated by four separate business lines from a large, international commercial bank. The P/L is based on the change in position values recorded at the close of each day and it does not include brokerage fees or commissions. Each series is constructed and defined in a consistent manner but the series are normalized to protect the bank’s anonymity.\(^1\)

For two of the business lines we have over 600 daily observations, whereas for the other two we have over 800 observations, yielding a panel of 2,930 observations. All four business lines are involved in securities trading but the exact nature of each business line is not known to us. We do know that there is very little overlap in assets across business lines. We also know that the different business lines are run by different employees and that all business lines rely on historical-simulation-based VaR systems for risk management. We do not observe the aggregate P&L summed across the business desks.

In addition to the daily revenue data, we obtained the corresponding one-day-ahead value-at-risk forecasts. The VaR forecasts are estimates of the 99% lower tail and are calculated for each business line within the bank. The bank relies on historical simulation for computing VaR.

Suppose revenue is denoted by \( R_t \). The \( p \% \) VaR is the quantity \( VaR_t \) such that

\[
F(R_{t+1} < VaR_t \mid \Omega_t) = p, \tag{1}
\]

\(^1\) The normalization that we employ does not imply that the P/L variance is one. However, the data is normalized by a constant and thus does not affect our results or the analysis in any way.
where $\Omega_i$ is the risk manager’s time $t$ information set. The VaR is the $p$th percentile of the return distribution. The probability $p$ is referred to as the coverage rate. By definition, the coverage rate is the probability that the lower tail VaR will be exceeded on a given day.

In our data set, the tail percentile of the bank’s VaR is set at $p = 0.01$, which yields a one-sided, 99% confidence interval. This is quite far in the tail but is typical of the VaR forecasts at commercial bank (e.g., Berkowitz and O’Brien 2002).

The daily P/L (dashed) and associated VaR (solid) are plotted over time in Figure 1. Business line 1 is observed from January 2, 2001, to June 30, 2004; business line 2 is observed from April 2, 2001; and business lines 3 and 4 are observed from January 3, 2002. Several interesting observations are apparent in Figure 1. First, notice that bursts of volatility are apparent in each of the P/L series (e.g., midsample for line 1 and end-sample for line 2), but these bursts are not necessarily synchronized across business lines. Second, note the occasional and very large spikes in the P/Ls. These are particularly evident for lines 1 and 2. Third, the bank VaRs exhibit considerable short-term variability (line 3), sometimes they show persistent trends away from the P/Ls (line 1) and even what looks like regime shifting without corresponding moves in the associated P/L (line 2). This can happen in a case where the bank took a large position on an asset that had volatile P/L in the recent past, thus not affecting the current business line’s P/L but increasing its historical simulation VaR, which is based on reconstructed—or pseudo—P/L series.

Table 1 reports the first four sample moments of the P/Ls and VaRs along with the exact number of daily observations. Of particular interest are the skewness and kurtosis estimates. Skewness is evident in business line 1 (negative) and business line 2 (positive) but much less so in business lines 3 and 4. Excess kurtosis is evident in all four business lines and dramatically so in lines 1 and 2. The skewness statistics confirm the occasional spikes in the P/Ls in Figure 1. For completeness, the descriptive statistics for theVaRs are also reported in Table 1.

The occasional bursts of volatility apparent in the P/Ls in Figure 1 are explored further in Figure 2, where we demean the P/Ls and plot their daily absolute values over time. Although the spikes in P/Ls dominate the pictures, episodes of high volatility are

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**Figure 1** P/Ls and One-day, 1% VaRs for Four Business Lines

**Figure 2** Absolute Demeaned P/Ls for Four Business Lines

**Table 1** P/Ls and VaRs for Four Business Lines: Descriptive Statistics

<table>
<thead>
<tr>
<th></th>
<th>Desk 1</th>
<th>Desk 2</th>
<th>Desk 3</th>
<th>Desk 4</th>
</tr>
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<tr>
<td>P/Ls</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of observations</td>
<td>873</td>
<td>811</td>
<td>623</td>
<td>623</td>
</tr>
<tr>
<td>Mean</td>
<td>0.1922</td>
<td>1.5578</td>
<td>1.8740</td>
<td>3.1562</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>2.6777</td>
<td>5.2536</td>
<td>1.6706</td>
<td>9.2443</td>
</tr>
<tr>
<td>Skewness</td>
<td>−1.7118</td>
<td>1.5441</td>
<td>0.5091</td>
<td>−0.1456</td>
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<tr>
<td>Excess kurtosis</td>
<td>24.2195</td>
<td>19.8604</td>
<td>2.0600</td>
<td>3.6882</td>
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<tr>
<td>VaRs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of observations</td>
<td>873</td>
<td>811</td>
<td>623</td>
<td>623</td>
</tr>
<tr>
<td>Mean</td>
<td>−7.2822</td>
<td>−16.3449</td>
<td>−3.2922</td>
<td>−24.8487</td>
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<tr>
<td>Standard deviation</td>
<td>3.1321</td>
<td>10.5446</td>
<td>1.1901</td>
<td>6.6729</td>
</tr>
<tr>
<td>Skewness</td>
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<td>−1.3746</td>
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<td>−0.3006</td>
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<tr>
<td>Kurtosis</td>
<td>−0.1525</td>
<td>1.6714</td>
<td>−0.0133</td>
<td>−0.1211</td>
</tr>
<tr>
<td>Observed number of hits</td>
<td>9</td>
<td>5</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Expected number of hits</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

**Notes.** We report various descriptive statistics for the daily P/Ls and daily 1%, one-day VaRs for each desk. The number of hits refers to the number of days on which the ex post loss exceeded the ex ante VaR.
4. A Unified Framework for VaR Evaluation

Under the 1996 Market Risk Amendment to the Basel Accord (Basel Committee on Banking Supervision 1996) effective in 1998, qualifying financial institutions have the freedom to specify their own model to compute their value at risk. It thus becomes crucially important for regulators to assess the quality of the models employed by assessing the forecast accuracy—a procedure known as “backtesting.” The nonregulatory uses of VaR presented in §2 also call for their accurate measurements.

The accuracy of a set of VaR forecasts can be assessed by viewing them as one-sided interval forecasts. A violation of the VaR, also called a “hit,” is defined as occurring when the ex post return is lower than the VaR. Specifically, we define violations as

$$ I_{t+1} = \begin{cases} 1 & \text{if } R_{t+1} < \text{VaR}_t(p), \\ 0 & \text{otherwise}, \end{cases} \quad (2) $$

i.e., a sequence of zeros and ones. By definition, the conditional probability of violating the VaR should always be

$$ \Pr(I_{t+1} = 1 \mid \Omega_t) = p \quad (3) $$

for every time $t$. The critical upshot is that no information available to the risk manager at the time the VaR was made should be helpful in forecasting the probability that the VaR will be exceeded. If it were, then this information should be incorporated into constructing a better VaR with unpredictable violations. We will refer to tests of this property as conditional coverage (CC) tests.

An unconditional coverage (UC) test of whether $\Pr(I_{t+1} = 1) = p$, under the assumption that the violations are independent, was developed in Kupiec (1995). The UC test rejects the null of an accurate VaR if the actual fraction of VaR violations in a sample is statistically different than $p$. We may expect Kupiec’s (1995) test to have lower power than other tests considered in our study because it cannot capture time series dependence in the violations.

4.1. Autocorrelation Tests

Christoffersen (1998) notes that property (3) implies that any sequence of violations, $(I_t)$, should be an independent and identically distributed (i.i.d.) Bernoulli random variable with mean $p$. To formally test this, Christoffersen (1998) embeds the null hypothesis of an i.i.d. Bernoulli within a general first-order Markov process.

If $(I_t)$ is a first-order Markov process, the one-step-ahead transition probabilities $pr(I_{t+1} \mid I_t)$ are given by

$$ \begin{bmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{bmatrix}, \quad (4) $$

where $\pi_{ij}$ is the transition $pr(I_{t+1} = j \mid I_t = i)$.

Under the null, the violations have a constant conditional mean that implies the two linear restrictions, $\pi_{01} = \pi_{11} = p$. A likelihood ratio test of these restrictions can be computed from the likelihood function

$$ L(I; \pi_{01}, \pi_{11}) = (1 - \pi_{01})^{i_0} \pi_{01}^{i_1} (1 - \pi_{11})^{i_1} \pi_{11}^{i_1}, \quad (7) $$

where $T_{ij}$ denotes the number of observations with a $j$ following an $i$, and $T_i$ is the number of $i$, i.e., is the number of ones or zeros in the sample.

We note that all of the tests we consider are carried out conditional on the value of the first observation. Although the tests all have known asymptotic distributions, we will rely on finite-sample $p$-values as discussed below.

In this paper, we extend and unify the existing tests by noting that the de-meaned violations $\{I_t - p\}$ form a martingale difference sequence (m.d.s.). By definition of the violation, Equations (2) and (3) immediately imply that

$$ E[(I_{t+1} - p) \mid \Omega_t] = 0, \quad (5) $$

where $\Omega_t$ is the information set of the risk manager up to time $t$. The de-meaned violations form an m.d.s. with respect to the time $t$ information set. This will be an extremely useful property because it implies that the violation sequence is uncorrelated at all leads and lags. For any variable $Z_t$ in the time $t$ information set, we then must have

$$ E[(I_{t+1} - p) \otimes Z_t] = 0, \quad (6) $$

which is familiar as the basis of generalized method of moments estimation.

This motivates a variety of tests that focus on the white noise or martingale property of the sequence. Because white noise has zero autocorrelations at all leads and lags, the violations can be tested by calculating statistics based on the sample autocorrelations.

Thus, specifying $Z_t$ to be the most recent de-meaned violation, we have

$$ E[(I_{t+1} - p)(I_t - p)] = 0. \quad (7) $$
The violation sequence has a first-order autocorrelation of zero, under the null. It is this property that is exploited by the Markov test of Christoffersen (1998). More generally, if we set $Z_t = I_{t-k}$ for any $k \geq 0$,

$$E[(I_{t+1} - p)(I_{t-k} - p)] = 0,$$

(8)

which says that the de-meaned violation sequence is in fact white noise. We write this null hypothesis compactly as

$$(l_{t+1} - p)_{\text{iid}} \sim (0, p(1-p)).$$

(9)

A natural testing strategy is to check whether any of the autocorrelations are not zero. Under the null, all the autocorrelations are zero,

$$H_0: \gamma_k = 0, \quad k > 0,$$

and the alternative hypothesis of interest is that

$$H_a: \gamma_k \neq 0, \quad \text{for some } k.$$

The Portmanteau or Ljung-Box statistics, for example, have a known distribution, which can be compared to critical values under the white noise null. The Ljung-Box statistic is a joint test of whether the first $m$ autocorrelations are zero. We can immediately make this into a test of a VaR model by calculating the autocorrelations of $(I_{t+1} - p)$ and then calculating

$$\text{LB}(m) = T(T+2) \sum_{k=1}^{m} \frac{\gamma_k^2}{T-k},$$

which is asymptotically chi-square with $m$ degrees of freedom.

We may also want to consider whether violations can be predicted by including other data in the risk manager’s information set such as past returns. Under the null hypothesis, it must be that

$$E[(I_{t+1} - p) g(I_t, I_{t-1}, \ldots, R_t, R_{t-1}, \ldots)] = 0$$

(10)

for any nonanticipating function $g(.)$. In analogy with Engle and Manganelli (2004), we might consider the $n$th-order autoregression

$$I_t = \alpha + \sum_{k=1}^{n} \beta_{1k} I_{t-k} + \sum_{k=1}^{n} \beta_{2k} g(I_{t-k}, I_{t-k-1}, \ldots, R_{t-k}, R_{t-k-1}, \ldots) + u_t,$$

(11)

where we set $g(I_{t-k}, I_{t-k-1}, \ldots, R_{t-k}, R_{t-k-1}, \ldots) = \text{VaR}_{t-k+1}$ and $n = 1$.

Estimating this autoregression by ordinary least squares would leave us having to deal with heteroskedasticity to make valid inference because the hit sequence is binary. We instead assume that the error term $u_t$ has a logistic distribution and we estimate a logit model. We can then test with a likelihood ratio test whether the $\beta$ coefficients are statistically significant and whether $\text{Pr}(I_t = 1) = e^{\alpha}/(1 + e^{\alpha}) = p$. We refer to this test as the CaViaR test of Engle and Manganelli (2004).

### 4.2. Hazard Rates and Tests for Clustering in Violations

Under the null that VaR forecasts are correctly specified, the violations should occur at random time intervals. Suppose the duration between two violations is defined as

$$D_t = t_i - t_{i-1},$$

(12)

where $t_i$ denotes the day of the violation number $i$. The duration between violations of the VaR should be completely unpredictable. There is an extensive literature on testing duration dependence (e.g., Kiefer 1988, Engle and Russel 1998, Gourieroux 2000), which makes this approach particularly attractive.

Christoffersen and Pelletier (2004) and Haas (2005) apply duration-based tests to the problem of assessing VaR forecast accuracy. In this section, we expand upon their methods. The duration-based tests can be viewed as another procedure for testing whether the violations form a martingale difference sequence.

Using the Bernoulli property, the probability of a violation next period is exactly equal to $\text{Pr}(D_i = 1) = \text{Pr}(I_{t+1} = 1) = p$. The probability of a violation in $d$ periods is

$$\text{Pr}(D_i = d) = \text{Pr}(I_{t+1} = 0, I_{t+2} = 0, \ldots, I_{t+d} = 1).$$

(13)

Under the null of an accurate VaR forecast, the violations are distributed

$$I_{t+1} \sim \text{iid}(p, p(1-p)).$$

This allows us to rewrite (13) as

$$\text{Pr}(D_i = d) = (1-p) \cdots (1-p)(p) = (1-p)^d p.$$

(14)

Equation (14) says that the density of the durations declines geometrically under the null hypothesis.

A more convenient representation of the same information is given by transforming the geometric probabilities into a flat function. The hazard rate defined as

$$\lambda(D_i) = \frac{\text{Pr}(D_i = d)}{1 - \text{Pr}(D_i < d)}$$

(15)

is such a transformation. Writing out the hazard function $\lambda(D_i)$ explicitly,

$$\frac{(1-p)^d p}{1 - \sum_{j=0}^{d-2} (1-p)^j} = p,$$

(16)

collapses to a constant after expanding and collecting terms.

We conclude that under the null, the hazard function of the durations should be flat and equal to $p$. Tests of this null are constructed by Christoffersen and Pelletier (2004). They consider alternative hypotheses under which the violation sequence, and hence the durations, display dependence or clustering. The
only (continuous) random distribution without duration dependence is the exponential; thus, under the null hypothesis the distribution of the durations should be

\[ f_{\text{exp}}(D; p) = pe^{-pD}. \]

The most powerful of the two alternative hypotheses they consider is that the durations follow a Weibull distribution where

\[ f_{W}(D; a, b) = a^b b^D D^{b-1} \exp(-(aD)^b). \]

This distribution is able to capture violation clustering. When \( b < 1 \), the hazard, i.e., the probability of getting a violation at time \( D_i \) given that we did not up to this point, is a decreasing function of \( D_i \).

It is also possible to capture duration dependence without resorting to the use of a continuous distribution. We can introduce duration dependence by having nonconstant probabilities of a violation,

\[
\Pr(D_i = d) = \Pr(I_{i+1} = 0, I_{i+2} = 0, \ldots, I_{i+d} = 1) \\
= (1 - p_1)(1 - p_2) \cdots (1 - p_{d-1})p_d,
\]

where

\[ p_d = \Pr(I_{i+d} = 1 | I_{i+d-1} = 0, \ldots, I_{i+1} = 0). \]

In this case, one must specify how these probabilities \( p_d \) vary with \( d \). We will set

\[ p_d = ad^{b-1} \]

with \( b \leq 1 \) to implement the test. We refer to this as the geometric test below.

Except for the first and last duration, the procedure is straightforward; we just count the number of days between each violation. We then define a binary variable \( C_i \), which tracks whether observation \( i \) is censored or not. Except for the first and last observation, we always have \( C_i = 0 \). For the first observation, if the hit sequence starts with zero, then \( D_1 \) is the number of days until we get the first hit. Accordingly \( C_1 = 1 \) because the observed duration is left-censored. The procedure is similar for the last observation. If the last observation of the hit sequence is zero, then the last duration, \( D_{N(T)} \), is the number of days after the last one in the hit sequence, and \( C_{N(T)} = 1 \) because the spell is right-censored.

The contribution to the likelihood of an uncensored observation is its corresponding probability density function (p.d.f.). For a censored observation, we merely know that the process lasted at least \( D_1 \) or \( D_{N(T)} \) days, so the contribution to the likelihood is not the p.d.f. but its survival function \( S(D_i) = 1 - F(D_i) \).

Combining the censored and uncensored observations, the log-likelihood is

\[
\ln L(D; a, b) = C_i \ln S(D_i) + (1 - C_i) \ln f(D_i) + \sum_{i=2}^{N(T)-1} \ln f(D_i) \\
+ C_{N(T)} \ln S(D_{N(T)}) + (1 - C_{N(T)}) \ln f(D_{N(T)}).
\]

Once the durations are computed and the truncations taken care of, then the likelihood ratio tests can be calculated in a straightforward fashion. The null and alternative hypotheses for the test are

\[
H_0: b = 1 \quad \text{and} \quad a = p; \\
H_a: b \neq 1 \quad \text{or} \quad a \neq p.
\]

The only added complication is that the maximum likelihood estimates are no longer available in closed form; they must be found using numerical optimization.

### 4.3. Spectral Density Tests

Another method for testing the martingale property is to examine the shape of the spectral density function. There is a long-standing literature on using the spectral density for this purpose because white noise has a particularly simple representation in the frequency domain—its spectrum is a flat line (e.g., Durlauf 1991). Statistical tests are constructed by examining whether the sample spectrum is “close” to the theoretical flat line.

The spectral density function is defined as a transformation of the autocovariance sequence,

\[
f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-\imath k \omega}. \tag{17}
\]

For a white noise process, all the autocovariances equal zero for any \( k \neq 0 \). This means that for the hit sequence the spectral density collapses to

\[
f(\omega) = \frac{1}{2\pi} p(1 - p) \tag{18}
\]

for all \( \omega \in [0, \pi] \).

The spectral density function is constant and proportional to the variance. Equivalently, the spectral distribution function is a 45° line. The asymptotic theory centers on the convergence of the random, estimated spectral density function using a functional central limit theorem.

The sample spectrum (or periodogram) is given by replacing the population autocovariances with the finite-sample estimates,

\[
\hat{f}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{\gamma}_k e^{-\imath k \omega}, \tag{19}
\]

which should be approximately a flat line.
It is often convenient to de-mean the sample spectral density and take the partial sums

$$\hat{U}(\omega) = \sum_{\omega=0}^{\pi} \left( \frac{\hat{f}(\omega)}{\hat{\sigma}^2} - \frac{1}{\pi} \right) \quad (20)$$

for each frequency $\omega \in [0, \pi]$. The $\hat{U}(\omega)$ are deviations of the sample spectral distribution from the 45th line. If the violations are white noise, the deviations should be small.

Durlauf (1991) derives the asymptotic distribution of a variety of statistics based on these deviations. The Cramér-Von Mises (CVM) test statistic is the sum of squared deviations

$$\text{CVM} = \sum_{\omega=0}^{\pi} \hat{U}(\omega)^2, \quad (21)$$

and it converges to a known distribution whose critical values can be tabulated numerically.

Another common test statistic dates to Bartlett (1955), who showed that the supremum

$$\sup_{\omega} \hat{U}(\omega)^2 \quad (22)$$

converges to the Kolmogorov-Smirnov (KS) statistic.

These test statistics have several attractive features. Unlike some tests of white noise (e.g., variance ratio tests), spectral tests have power against any linear alternative of any order. That is, the test has power to detect any second moment dynamics (see Durlauf 1991). Both the CVM and KS statistics diverge asymptotically if $I_i$ is any stationary process that is not white noise.

4.4. Multivariate Tests

The tests described above only use information about one hit sequence at a time. In a case where we have values at risk and P/L for different business lines, we might be interested in jointly testing if property (3) holds for all the hit sequences. In this way, we could hope that the tests would have more power because we would be effectively increasing the sample size.

A first approach to study simultaneously the hit sequences could be to simply “stack” the series together, assuming that the series are independent across desks (separate realizations from the same process). For the Ljung-Box test, we could compute the autocorrelations using all the series, treating them as multiple nonoverlapping sequences from the same underlying process. For likelihood-based tests such as the duration tests in §4.2, we could sum the log-likelihoods for each series. All of the above are based on a likely unrealistic independence assumption.

A second approach would be to capture the dependence across the series by considering multivariate generalizations of the previous tests. Recall from Equation (3) that no information available to the risk manager at the time the VaR is made should be helpful in forecasting a VaR violation. Thus, if the VaR models are correctly specified, then past observations from the hit sequence of one business line, which are clearly available to the risk manager, should not help predict violations of another business line. One could then consider using multivariate Box-Pierce tests as in Lütkepohl (1993, §4.4), or multivariate spectral tests as in Paramasamy (1992). Duration-based tests could be extended by considering competing risk models following Cameron and Trivedi (2005, Chap. 19). Perhaps the easiest way to use information from all of the business lines is offered by the regression approach of the CaViaR test. We can simply use variables from other business lines, such as their P/Ls, as explanatory variables. The conditional coverage test would then consist in testing that the coefficients of the explanatory variables (such as P/Ls) are zero and the probability of getting a violation is equal to $p$. For the Kupiec test, a multivariate version of the unconditional coverage test is developed in Perignon and Smith (2007).

5. Size and Power Properties

Given the large variety of backtesting procedures surveyed in §3, it is important to give risk managers guidance as to their comparative size and power properties in a controlled setting.

5.1. Effective Size of the Tests

To assess the size properties of the various methods, we simulate i.i.d. Bernoulli samples with probabilities $p = 1\%$ and $5\%$. For each Bernoulli probability, we consider several different sample sizes, from 250 to 1,500. Rejection rates under the null are calculated over 10,000 Monte Carlo trials. If the asymptotic distribution is accurate in the sample sizes considered, then the rejection frequencies should be close to the nominal size of the test, which we set to 10%. In the CaViaR test, we generate the required VaR regressors via a generalized autoregressive conditional heteroskedasticity (GARCH) model with innovations that are independent of the simulated hit sequence. This way we perform a test that is true to the CaViaR idea while ensuring that the null hypothesis is true.

Table 2 contains the actual size of the CC tests when the asymptotic critical values are used. The number of observations in each simulated sample is reported in the first column. The top panel shows the finite-sample test sizes for a $1\%$ VaR. We see that the LB(1) test tends to be undersized and the LB(5) oversized in finite samples. The Markov is somewhat undersized and the Weibull test oversized. The geometric test is extremely oversized for the smallest sample. The CaViaR test is undersized. The CVM test is undersized...
for the smallest sample size and oversized for the larger samples. Finally, the Kupiec (1995) unconditional test and the KS test have good size properties beyond the smallest sample sizes.

The results in the bottom panel cover the 5% VaR. In this case, the LB(1) test is slightly undersized, whereas the LB(5) is very close to the desired 10%. The Markov and Weibull tests are both oversized. The geometric test is somewhat undersized, whereas the Kupiec, CaViaR, KS, and CVM tests now are very close to the desired 10% level.

The overall conclusion from Table 2 is that for small sample sizes and for the 1% VaR, which is arguably the most common in practice, the asymptotic critical values can be highly misleading. When computing power below we therefore rely on the Dufour (2006) Monte Carlo testing technique, which is described in detail in §6.

### 5.2. Finite-Sample Power of the Tests

To perform a power comparison, we use a flexible and simple GARCH specification as a model of the P/L process. GARCH models are some of the most widely used models for capturing variance dynamics in daily asset returns. See Andersen et al. (2006) for a recent survey. We estimate the parameters for each business line separately to model the volatility persistence in each series.

The GARCH model allows for an asymmetric volatility response or “leverage effect.” In particular, we use the NGARCH(1, 1)-t(d) specification,

\[
R_{t+1} = \sigma_{t+1} ((d - 2)/d)^{1/2} z_{t+1},
\]

\[
\sigma_{t+1}^2 = \omega + \alpha \sigma_t^2 + \beta ((d - 2)/d) z_t - \theta)^2 + \beta \sigma_t^2,
\]

where \(R_{t+1}\) is the daily demeaned P/L and the innovations \(z_t\) are drawn independently from a Student’s \(t(d)\) distribution. The Student’s \(t\) innovations enable the model to capture some of the additional kurtosis.

Table 3 reports the maximum likelihood estimates from the GARCH model for each business line. As usual, we get a small but positive \(\alpha\) and \(\beta\) much closer to one. Variance persistence in this model is given by \(\alpha(1 + \theta^2) + \beta\). It is largest in business lines 2 and 4, which confirms the impression provided by Figure 2. The last three lines of Table 3 report the log-likelihood values for the four GARCH models along with the log-likelihood values for the case of no variance dynamics, where \(\alpha = \beta = \theta = 0\).

Looking across the four GARCH estimates we see that Desk 1 is characterized by a large \(\omega\) and small \(d\), which suggests larger kurtosis. Desk 2 is characterized by high variance persistence and high unconditional kurtosis from the low \(d\). Desk 3 has an unusually large negative \(\theta\), which suggests that a positive P/L

### Table 2 Size of 10% Asymptotic CC Tests

<table>
<thead>
<tr>
<th>Sample</th>
<th>LB(1)</th>
<th>LB(5)</th>
<th>Markov</th>
<th>Weibull</th>
<th>Geometric</th>
<th>CaViaR</th>
<th>KS</th>
<th>CVM</th>
<th>Kupiec</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>0.025</td>
<td>0.100</td>
<td>0.050</td>
<td>0.110</td>
<td>0.531</td>
<td>0.046</td>
<td>0.039</td>
<td>0.052</td>
<td>0.122</td>
</tr>
<tr>
<td>500</td>
<td>0.044</td>
<td>0.134</td>
<td>0.068</td>
<td>0.176</td>
<td>0.233</td>
<td>0.069</td>
<td>0.066</td>
<td>0.112</td>
<td>0.068</td>
</tr>
<tr>
<td>750</td>
<td>0.067</td>
<td>0.165</td>
<td>0.066</td>
<td>0.162</td>
<td>0.158</td>
<td>0.070</td>
<td>0.092</td>
<td>0.124</td>
<td>0.100</td>
</tr>
<tr>
<td>1,000</td>
<td>0.076</td>
<td>0.147</td>
<td>0.076</td>
<td>0.157</td>
<td>0.119</td>
<td>0.067</td>
<td>0.094</td>
<td>0.125</td>
<td>0.117</td>
</tr>
<tr>
<td>1,250</td>
<td>0.102</td>
<td>0.146</td>
<td>0.055</td>
<td>0.128</td>
<td>0.111</td>
<td>0.075</td>
<td>0.112</td>
<td>0.140</td>
<td>0.116</td>
</tr>
<tr>
<td>1,500</td>
<td>0.101</td>
<td>0.131</td>
<td>0.064</td>
<td>0.127</td>
<td>0.095</td>
<td>0.071</td>
<td>0.112</td>
<td>0.137</td>
<td>0.121</td>
</tr>
</tbody>
</table>

Notes. We simulate i.i.d. Bernoulli variables to assess the size properties of the various asymptotic backtesting procedures. LB(1) and LB(5) are Ljung-Box tests with 1 and 5 lags, respectively. Markov is a first-order Markov test. Weibull and geometric are duration-based tests. CaViaR is a regression-based test. Kupiec is a test of unconditional coverage. Please see the text for details on each test.

### Table 3 P/L GARCH Model Parameters and Properties

<table>
<thead>
<tr>
<th>Desk</th>
<th>(d)</th>
<th>(\theta)</th>
<th>(\beta)</th>
<th>(\alpha)</th>
<th>(\omega)</th>
<th>Variance persistence</th>
<th>Unconditional std. dev.</th>
<th>logL</th>
<th>logL (homosked.)</th>
<th>(p)-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.808</td>
<td>0.245</td>
<td>0.7495</td>
<td>0.1552</td>
<td>0.5469</td>
<td>0.9140</td>
<td>2.5220</td>
<td>-1.3607</td>
<td>-1.4016</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>3.3183</td>
<td>0.5031</td>
<td>0.9284</td>
<td>0.0524</td>
<td>0.2154</td>
<td>0.9941</td>
<td>6.0233</td>
<td>-1.7812</td>
<td>-1.8434</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>6.9117</td>
<td>-0.9616</td>
<td>0.8728</td>
<td>0.0261</td>
<td>0.2127</td>
<td>0.9230</td>
<td>1.6624</td>
<td>-825.87</td>
<td>-831.46</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>4.7017</td>
<td>0.0928</td>
<td>0.9153</td>
<td>0.0723</td>
<td>1.6532</td>
<td>0.9882</td>
<td>11.8478</td>
<td>-1.8555</td>
<td>-1.8777</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Notes. Using the maximum likelihood, we estimate on each desk P/L an asymmetric GARCH(1,1) model with standardized Student’s \(t(d)\)-distributed innovations. The \(p\)-value reports the significance level of a test of homoskedastic \(t(d)\) returns against the heteroskedastic GARCH-\(t(d)\) alternative.
increases volatility by more than a negative P/L of the same magnitude. Desk 4 has an unusually large unconditional volatility and a relatively high persistence as noted earlier. Overall, our GARCH estimates are similar to ones obtained by Perignon and Smith (2007) with aggregate bank data, but our estimates of the Student’s t(4) degree of freedom are in the lower range of the usual values obtained with various financial returns.

For the power simulation exercise, we will assume that the correct data-generating processes are the four estimated GARCH processes. We must also choose a particular implementation for the VaR calculation. Following industry practice (see Perignon and Smith 2008) and the approach used by the bank that provided us with the VaR data in Figure 1, we rely on historical simulation or “bootstrapping.” The historical simulation VaR on a certain day is simply the unconditional quantile of the past \( t \) daily observations. Specifically,

\[
\text{VaR}_t^p = \text{percentile} \left( \{ R_s \}_{s=T-t+1}^{T} , 100p \right).
\]

For the purposes of this Monte Carlo experiment, we choose \( T = 250 \) corresponding to 250 trading days.

The VaR coverage rate \( p \) we study is either 1% (as in §3) or 5%. We look at the one-day-ahead VaR again as in §3. When computing the finite-sample \( p \)-values we use 9,999 simulations, and we perform 10,000 Monte Carlo simulations for each test. Section 6 provides the details of the \( p \)-value simulation.

Table 4 shows the finite-sample power results, based on a 10% significance level, for the 1% VaR from historical simulation for various samples sizes when using the GARCH data-generating processes corresponding to each of the four business lines.

For all of the sample sizes in all four business lines in Table 4, the CaViaR test performs the best. For business line 1, the LB(5), the KS, and the CVM tests perform well also. For business line 2, the geometric test also performs well. For business line 3, only the CaViaR test has good power. For business line 4, the LB(5) and the KS tests perform well in addition to the CaViaR test.

Consider next Table 5, which shows reports the finite-sample power calculations for the 5% VaR. For business line 1, the LB(5) and the CaViaR are best. For business line 2, the CaViaR test is best for small samples but the geometric test is best for the larger sample sizes we examine. For business line 3, the power
Tables 4 and 5 provide a couple of other conclusions. First, it is clear that the LB(5) test is better than LB(1) and Markov test. This is perhaps to be expected because the dependence in the hit sequence is not of a first-order here. Second, the geometric test is substantially better than the Weibull test. This is also to be expected as the latter wrongly assumes a continuous distribution for the duration variable.

Overall, the power of the best conditional tests is quite impressive. The CaViaR tests display strong power to reject inaccurate VaR, especially compared to the unconditional test. This is important because regulatory capital includes a penalty if the unconditional number of exceptions is too high, so the charge for an inaccurate VaR is implicitly dependent on an unconditional test. No formal backtesting method is currently recommended under the Basel Accord, but the evidence presented here strongly suggests a method along the lines of the CaViaR method rather than a method based on the unconditional violation rate.

### 5.3. Feasibility Ratios

For transparency we report in Table 6 the fraction of simulated samples from Tables 4 and 5 where each test is feasible. We only report sample sizes 250, 500, and 750 for the 1% VaR and 250 for the 5% VaR.

<table>
<thead>
<tr>
<th>Sample</th>
<th>LB(1)</th>
<th>LB(5)</th>
<th>Markov</th>
<th>Weibull</th>
<th>Geometric</th>
<th>CaViaR</th>
<th>KS</th>
<th>CVM</th>
<th>Kupiec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Business line 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>0.296</td>
<td>0.385</td>
<td>0.205</td>
<td>0.161</td>
<td>0.319</td>
<td>0.447</td>
<td>0.349</td>
<td>0.344</td>
<td>0.157</td>
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<td>0.391</td>
<td>0.528</td>
<td>0.214</td>
<td>0.183</td>
<td>0.447</td>
<td>0.517</td>
<td>0.443</td>
<td>0.464</td>
<td>0.069</td>
</tr>
<tr>
<td>750</td>
<td>0.436</td>
<td>0.633</td>
<td>0.226</td>
<td>0.231</td>
<td>0.568</td>
<td>0.611</td>
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<td>0.553</td>
<td>0.033</td>
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<td>0.270</td>
<td>0.679</td>
<td>0.692</td>
<td>0.586</td>
<td>0.607</td>
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<td>0.815</td>
<td>0.720</td>
<td>0.722</td>
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<td>Business line 2</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>0.259</td>
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<td>0.340</td>
<td>0.322</td>
<td>0.422</td>
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<td>0.298</td>
<td>0.366</td>
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<td>0.617</td>
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<td>0.449</td>
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<td>750</td>
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</tr>
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<td>0.819</td>
<td>0.681</td>
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<td>Business line 3</td>
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<tr>
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<td>0.108</td>
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<td>0.068</td>
<td>0.089</td>
<td>0.299</td>
<td>0.099</td>
<td>0.099</td>
<td>0.057</td>
</tr>
<tr>
<td>500</td>
<td>0.103</td>
<td>0.120</td>
<td>0.065</td>
<td>0.040</td>
<td>0.064</td>
<td>0.351</td>
<td>0.105</td>
<td>0.112</td>
<td>0.014</td>
</tr>
<tr>
<td>750</td>
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<td>0.062</td>
<td>0.034</td>
<td>0.049</td>
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<td>0.001</td>
</tr>
<tr>
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<td>0.053</td>
<td>0.713</td>
<td>0.117</td>
<td>0.116</td>
<td>0.001</td>
</tr>
<tr>
<td>Business line 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>0.288</td>
<td>0.393</td>
<td>0.331</td>
<td>0.326</td>
<td>0.446</td>
<td>0.590</td>
<td>0.409</td>
<td>0.393</td>
<td>0.353</td>
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<tr>
<td>500</td>
<td>0.353</td>
<td>0.536</td>
<td>0.282</td>
<td>0.386</td>
<td>0.626</td>
<td>0.625</td>
<td>0.470</td>
<td>0.474</td>
<td>0.235</td>
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<tr>
<td>750</td>
<td>0.398</td>
<td>0.637</td>
<td>0.267</td>
<td>0.465</td>
<td>0.746</td>
<td>0.672</td>
<td>0.545</td>
<td>0.540</td>
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<td>1,000</td>
<td>0.443</td>
<td>0.705</td>
<td>0.278</td>
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<td>0.729</td>
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<td>1,500</td>
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<td>0.837</td>
<td>0.734</td>
<td>0.713</td>
<td>0.132</td>
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</tbody>
</table>

Notes. We simulate hit sequences from GARCH P/Ls and historical simulation VaRs to assess the power properties of the tests. LB(1) and LB(5) are Ljung-Box tests with 1 and 5 lags, respectively. Markov is a first-order Markov test. Weibull and geometric are duration-based tests. CaViaR is a regression-based test. Kupiec is a test of unconditional coverage. Please see the text for details on each test.
6. Results for Desk-Level Data

In Table 7, we report the results from applying our tests to the actual observed sequences of P/Ls and historical simulation VaRs from the four business lines. As in the power calculations above, we make use of the Dufour (2006) Monte Carlo testing technique, which yields tests with correct level, regardless of sample size.

For the case of a continuous test statistic, the procedure is as follows: We first generate \( N \) independent realizations of the test statistic, \( LR_i, i = 1, \ldots, N \). We denote by \( LR_0 \) the test statistic computed with the original sample. Under the hypothesis that the risk model is correct, we know that the hit sequence is an i.i.d. Bernoulli with the mean equal to the coverage rate. We thus benefit from the advantage of not having nuisance parameters under the null hypothesis.

We next rank \( LR_i, i = 0, 1, \ldots, N \) in nondecreasing order and obtain the Monte Carlo \( p \)-value \( \hat{p}_N(LR_0) \), where

\[
\hat{p}_N(LR_0) = \frac{N \hat{G}(LR_0) + 1}{N + 1}
\]

and

\[
\hat{G}_N(LR_0) = \frac{1}{N} \sum_{i=1}^{N} I(LR_i > LR_0).
\]

The indicator function \( I(\cdot) \) takes on the value one if true and the value zero otherwise. We reject the null hypothesis if \( \hat{p}_N(LR_0) \) is less or equal than the prespecified significance level.

When working with binary sequences, there is a nonzero probability of obtaining ties between the test values obtained with the sample and the simulated data. The tiebreaking procedure is as follows. For each test statistic, \( LR_i, i = 0, 1, \ldots, N \), we draw an independent realization of a uniform distribution on the \([0, 1]\) interval. Denote these draws by \( U_i, i = 0, 1, \ldots, N \). We

<p>| Table 6 Fraction of Samples Where Test Is Feasible (1% and 5% VaR) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>VaR (%)</th>
<th>Sample</th>
<th>LB(1)</th>
<th>LB(5)</th>
<th>Markov</th>
<th>Weibull</th>
<th>Geometric</th>
<th>CaViaR</th>
<th>KS</th>
<th>CVM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power simulation: Business line 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>1</td>
<td>250</td>
<td>0.9081</td>
<td>0.9081</td>
<td>0.9006</td>
<td>0.6974</td>
<td>0.8322</td>
<td>0.8998</td>
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<tr>
<td>1</td>
<td>500</td>
<td>0.9984</td>
<td>0.9984</td>
<td>0.9974</td>
<td>0.9852</td>
<td>0.9918</td>
<td>0.9974</td>
<td>0.9983</td>
<td>0.9979</td>
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<tr>
<td>1</td>
<td>750</td>
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<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>5</td>
<td>250</td>
<td>0.9998</td>
<td>0.9998</td>
<td>0.9998</td>
<td>0.9998</td>
<td>0.9998</td>
<td>1.0000</td>
<td>0.9996</td>
<td>0.9999</td>
</tr>
<tr>
<td>Power simulation: Business line 2</td>
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<td></td>
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</tr>
<tr>
<td>1</td>
<td>250</td>
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<td>0.8693</td>
<td>0.8643</td>
<td>0.6691</td>
<td>0.8645</td>
<td>0.8693</td>
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<td>1</td>
<td>500</td>
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<td>0.9946</td>
<td>0.9928</td>
<td>0.9654</td>
<td>0.9924</td>
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<td>0.9946</td>
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<td>750</td>
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<td>0.9996</td>
<td>0.9999</td>
<td>0.9986</td>
<td>0.9996</td>
<td>0.9997</td>
<td>0.9997</td>
<td>0.9997</td>
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<td>5</td>
<td>250</td>
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<td>0.9965</td>
<td>0.9949</td>
<td>0.9881</td>
<td>0.9942</td>
<td>0.9958</td>
<td>0.9963</td>
<td>0.9973</td>
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<td>Power simulation: Business line 3</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>250</td>
<td>0.8615</td>
<td>0.8615</td>
<td>0.8660</td>
<td>0.6775</td>
<td>0.8645</td>
<td>0.8615</td>
<td>0.8615</td>
<td>0.8615</td>
</tr>
<tr>
<td>1</td>
<td>500</td>
<td>0.9935</td>
<td>0.9935</td>
<td>0.9940</td>
<td>0.9694</td>
<td>0.9939</td>
<td>0.9944</td>
<td>0.9941</td>
<td>0.9946</td>
</tr>
<tr>
<td>1</td>
<td>750</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9989</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
</tr>
<tr>
<td>5</td>
<td>250</td>
<td>0.9974</td>
<td>0.9974</td>
<td>0.9971</td>
<td>0.9895</td>
<td>0.9938</td>
<td>0.9972</td>
<td>0.9963</td>
<td>0.9957</td>
</tr>
<tr>
<td>Power simulation: Business line 4</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>1</td>
<td>250</td>
<td>0.9190</td>
<td>0.9190</td>
<td>0.9190</td>
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<td>0.7119</td>
<td>0.1910</td>
<td>0.1910</td>
<td>0.1910</td>
</tr>
<tr>
<td>1</td>
<td>500</td>
<td>0.9937</td>
<td>0.9937</td>
<td>0.9937</td>
<td>0.9619</td>
<td>0.9664</td>
<td>0.9938</td>
<td>0.9937</td>
<td>0.9937</td>
</tr>
<tr>
<td>1</td>
<td>750</td>
<td>0.9992</td>
<td>0.9992</td>
<td>0.9992</td>
<td>0.9949</td>
<td>0.9964</td>
<td>0.9994</td>
<td>0.9992</td>
<td>0.9992</td>
</tr>
<tr>
<td>5</td>
<td>250</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Notes. We report the fraction of simulations where the hit sequence allowed us to compute the test statistic. LB(1) and LB(5) are Ljung-Box tests with 1 and 5 lags, respectively. Markov is a first-order Markov test. Weibull and geometric are duration-based tests. CaViaR is a regression-based test. Please see the text for details on each test.

because the other sample sizes had no omitted sample paths in our experiment. Table 4 shows that only in the case of 1% VaR and samples of 250 observations is the issue nontrivial. In those cases, the issue is most serious for the Weibull and geometric tests. That conclusion also holds when considering the bottom panel in Table 6, which reports the fraction of feasible samples from the size calculations in Table 2. We do not report results for the Kupiec test because it can always be computed.
Table 7  Backtesting Actual VaRs from Four Business Lines

<table>
<thead>
<tr>
<th>LB(1)</th>
<th>LB(5)</th>
<th>Markov</th>
<th>Weibull</th>
<th>Geometric</th>
<th>CaViaR</th>
<th>VIX</th>
<th>KS</th>
<th>CVM</th>
<th>CavMult</th>
<th>Kupiec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test value</td>
<td>0.096</td>
<td>0.483</td>
<td>0.196</td>
<td>1.014</td>
<td>1.200</td>
<td>3.227</td>
<td>0.521</td>
<td>18.748</td>
<td>2.438</td>
<td>3.721</td>
</tr>
<tr>
<td>p-value</td>
<td>0.460</td>
<td>0.551</td>
<td>0.963</td>
<td>0.662</td>
<td>0.376</td>
<td>0.278</td>
<td>0.832</td>
<td>0.324</td>
<td>0.395</td>
<td>0.614</td>
</tr>
</tbody>
</table>

| Test value | 0.032 | 0.159 | 1.458 | 3.634 | 3.838 | 4.856 | 2.187 | 11.500 | 1.350 | 12.462 | 1.395 |
| p-value   | 0.825 | 0.838 | 0.320 | 0.235 | 0.125 | 0.131 | 0.411 | 0.467 | 0.562 | 0.040 | 0.259 |

| Test value | 0.002 | 0.008 | 6.849 | NaN | NaN | 7.561 | 27.303 | 70.365 | 70.365 | 9.800 | 6.846 |
| p-value   | 0.992 | 0.992 | 0.018 | NaN | NaN | 0.033 | 0.014 | 0.053 | 0.024 | 0.103 | 0.009 |

| Test value | 0.026 | 38.572 | 0.975 | 4.424 | 4.997 | 4.104 | 3.430 | 19.627 | 5.236 | 9.352 | 0.923 |
| p-value   | 0.785 | 0.009 | 0.369 | 0.172 | 0.060 | 0.177 | 0.284 | 0.182 | 0.180 | 0.118 | 0.330 |

Notes. We report the test statistics using the hit sequences from the actual P/Ls and VaRs from the four business lines. LB(1) and LB(5) are Ljung-Box tests with 1 and 5 lags, respectively. Markov is a first-order Markov test. Weibull and geometric are duration-based tests. CaViaR is a regression-based test. Kupiec is a test of unconditional coverage. The CavMult test uses the ex ante VaR from all four business lines in a CaViaR test. The VIX column is a CaViaR test where we use the value of the Chicago Board Options Exchange Volatility Index (VIX) as an explanatory variable instead of the VaR. NaN is to denote that the test cannot be computed. The bold numbers are p-values less than 10%. Please see the text for details on each test.

Thus, when backtesting actual VaRs, we reject their statistical accuracy for three out of four business lines. For business line 1, the VaR based on a 250-day historical simulation approach appears to work well. The same cannot be said for the other three business lines. Our backtests indicate that the VaR models for business lines 2–4 are statistically inaccurate and may need modification.

Our data set also allow us to explore how well this bank is able to forecast their portfolio risk at the business-line level. In the spirit of the Mincer and Zarnowitz (1969) method used in the forecasting literature, we can regress the absolute value of the P/L on the corresponding VaR. From Taylor (2005), we know that the VaR should be proportional to the standard deviation, so the R^2 from this regression is an indication of how much of the variability in the P/L could be forecasted by the computation of the VaR. In Table 8, we present three sets of R^2 values for each business line. The first R^2 value is the one obtained when regressing the business line’s absolute P/L on its VaR (computed with historical simulation) plus an intercept. To help us assess how big the first R^2 is, we simulate absolute P/L data from the GARCH models used in §5, and we report the R^2 for the following two regression of absolute P/Ls on the 1% one-day-ahead VaR obtained with either (i) historical simulation with a 250-day rolling window or (ii) the true VaR. The first number is the R^2 we would expect to obtain with the true data, and the second is an indication of the upper bound we could obtain.

Our result suggest that at the business-line level the bank forecasts risk as well as, if not better than, we would expect given the historical simulation method used to compute VaR. For three of out four business lines, the R^2 obtained with the real data is significantly higher than the one obtained with simulated...
data and historical simulation. But these $R^2$ values are much smaller than the ones where we use the true GARCH-based VaR in the regression (except for business line 3 where GARCH may fit poorly), indicating that the bank’s risk management system could quite likely be improved by incorporating dynamic volatility into the VaR computations.

7. Conclusions

The uses of VaR in banking are many and varied. All VaR applications share, however, the need for constant evaluation of the accuracy of the VaR risk measures reported. This is true regardless of whether the VaR is used in a passive or active way, and whether it is used in internal operations or externally for regulatory purposes.

The widespread and sudden losses experienced by financial services firms during the 1998 “currency crisis,” the 2000–2001 internet bubble, and the current collapse of collateralized debt securities, all serve to highlight the importance of making sure that risk measures are accurately calculated. Although having accurate VaR measures may not prevent volatility, accurate VaRs can be used to calculate risk levels and the appropriate amount of safe capital. Similarly, VaR measures cannot prevent traders from experiencing losses, but they can provide management with a sense of how risky their traders are behaving, and VaR-based trading limits can be instituted to control overall risk.

Using new desk-level P/Ls from four business lines in a large international commercial bank, we find evidence of volatility dynamics and nonnormality in the desk-level data. Volatility dynamics are not captured in historical simulation and may therefore cause clustering in VaR violations.

Formal backtesting techniques show the clustering is severe enough that we can reject the accuracy of the VaR models for two of the four business lines. A third business line VaR is rejected by the Kupiec test of unconditional coverage. This suggests that the set of VaR problems discussed here can successfully be detected by external bank regulators and internal risk auditors in real-world situations. Because no formal backtesting method is currently recommended under the Basel Accord, the evidence presented here strongly suggests a possible direction for improvements to future regulatory schemes. Regulators may benefit from adopting an approach along the lines of the CaViaR method rather than a method based on the unconditional violation rate.

Acknowledgments

The second author acknowledges financial support from CREATE, FQRSC, IFM2, and SSRHC, and the third author acknowledges financial support from the North Carolina State University Enterprise Risk Management Initiative and the Edwin Gill Research Grant.

References


